Electroweak Theory
and
Noncommutative Geometry

A. Çatal-Özer* and T. Dereli†

Department of Physics, Middle East Technical University
06531 Ankara, Turkey

Abstract

The noncommutative generalisation of the standard electroweak model due to Balakrishna, Gürsey and Wali is formulated in terms of the derivations $Der_2(M_3)$ of a three dimensional representation of the su(2) Lie algebra of weak isospin. A light Higgs boson of mass about 130 GeV, together with four very heavy scalar bosons are predicted.

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*E.Mail: acatal@newton.physics.metu.edu.tr
†E.Mail: tdereli@metu.edu.tr

1 Introduction

The standard electroweak theory is one of the most remarkable achievements of physics. In spite of its many successes it still cannot be considered complete because the fundamental scalar field whose existence is essential to the theory hasn't been observed. The elusive Higgs particle is a direct consequence of the mass generation mechanism that is induced by a quartic scalar self-interaction potential with arbitrary coupling parameters. Neither the strength of these couplings, nor the mass of the Higgs boson can be determined within the standard model. Therefore it is still worthwhile to look for a deeper understanding of the related phenomena. A popular approach to this problem involves introducing higher dimensional space-times with product topology so that the compact extra dimensions may be related to the internal symmetries of constituent particles. In this picture the microstructure of spacetime itself at the Planck scales becomes a subject of speculations. Several years ago a program based on noncommutative geometries was started by A. Connes [1]. The Connes-Lott noncommutative electroweak model [2] and its later elaborations [3] assume essentially a four dimensional, two-sheeted space-time and the Higgs scalars that are governed by a discrete $\mathbf{Z}_2$ symmetry relate different space-time sheets. A little later Connes himself [4] introduced a notion of reality that helps to eliminate most but not all unpleasant features of the original model. There is a recent book [5] that may be consulted for further literature.

The electroweak model based on a noncommutative geometry we study here has a somewhat different structure as the space-time degrees of freedom are extended by matrices, which are supposed to describe the internal symmetries of elementary particles. It was introduced about ten years ago by Balakrishna, Gürsey and Wali [6]. In this scheme, the Higgs scalars arise naturally with the correct weak charge assignments along with the gauge potentials as part of the connection and give rise in the Yang-Mills action functional to a quartic Higgs potential that appears already shifted to a spontaneously broken symmetric phase. This built-in mass generation mechanism involves a single dimensionless coupling strength and two largely separated mass scales. The subsequent renormalization group flow from beyond Planck scales down to the electroweak scale provides realistic values for the Weinberg mixing angle, the masses of the weak intermediate bosons as well as the mass of a light Higgs boson. The Higgs boson mass prediction of approximately $130GeV/c^2$ comes remarkably close to the energy range that generated much recent excitement among the Higgs searchers at CERN [7].

We give below a short survey of the noncommutative structure of the model and provide an analysis of the bosonic mass spectrum. More details are found in our previous paper [8].

2 Generalities on Matrix Geometries

The basic data needed in noncommutative geometry is the spectral triple or the K-cycle $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ were $\mathcal{A}$ is an involutive algebra of operators in Hilbert space $\mathcal{H}$ and $\mathcal{D}$ is a self-adjoint, unbounded operator in $\mathcal{H}$. In the commutative case the spectral triple corresponds to the usual arena of doing quantum field theory, namely, $\mathcal{A} = C^{\infty}(V)$ the algebra of smooth functions on the space-time manifold V, $\mathcal{H} = L^2(V, S)$ the Hilbert space of L^2-spinors, and $\mathcal{D}$ the Dirac operator of the Levi-Civita spin connection. Then a differential algebra from this spectral data is constructed so that generalized connections and curvatures can be defined. We should be able to follow a similar strategy in the noncommutative case. To do that we first of all notice that the group $Aut(\mathcal{A})$ of automorphisms of $\mathcal{A}$ is isomorphic to the group $Diff(V)$ of diffeomorphisms of V in the usual Riemannian case. The algebra describing the manifold is invariant under automorphisms. Since the algebra is noncommutative, the group $Aut(\mathcal{A})$ has a natural subgroup $Inn(\mathcal{A})$. An automorphism f is inner if and only if it acts as conjugation of the elements of the algebra by some unitary element $u \in \mathcal{A}$, that is $f(a) = uau^*, \forall a \in \mathcal{A}$. All the other automorphisms are called outer automorphisms and we can write $Aut(\mathcal{A}) = Inn(\mathcal{A}) \otimes Out(\mathcal{A})$. In the noncommutative formulation of the standard model inner and outer automorphisms appear as the transformations on the space-time and the internal space, respectively. Hence the coordinate transformations on space-time get unified with gauge transformations. We will extend the space-time degrees of freedom by matrices in which case the underlying C^*-algebra would be $\mathcal{A} = C^{\infty}(V) \otimes M_n(\mathbf{C})$ where $M_n(\mathbf{C})$ is the algebra of $n \times n$ matrices. This simple tensor product space is sufficient for our purpose of studying the bosonic sector of the electroweak theory. The formalism developed here should be considered as a particular case of a general theory and in this sense, it is more appropriate to call it "matrix geometry", rather than "noncommutative geometry" (see e.g. [5]).

To construct the action we need four ingredients: differential forms on $\mathcal{A}$, a Lie group G of "internal symmetries", a scalar product on the space of differential forms $\Omega^{\star}(\mathcal{A})$ and an invariant scalar product on the Lie algebra $\mathcal{G}$ of the group G. The scalar product we introduce on $\Omega^{\star}(\mathcal{A})$ will be

the substitute of the Dixmier trace. In fact a manifold V can be completely characterized by the commutative C^*-algebra of continuous, complex-valued functions on it, $C^\infty(V)$. In this matrix geometry version of noncommutative geometry, our noncommutative manifold can be completely characterized by replacing $C^\infty(V)$ by the algebra $M_n(\mathbf{C})$ which is again a C^*-algebra. The Lie algebra of complex vector fields coincides with the Lie algebra of derivations $Der(C^\infty(V))$. Similarly one can construct the differential algebra of M_n from the vector space $Der(M_n)$, space of all derivations of M_n.

3 Balakrishna-Gürsey-Wali Model

A choice of derivations determines the model and the choice we make is dictated by which symmetries we want unbroken at the end. We use the Lie subalgebra $Der_2(M_3)$ generated by a three dimensional representation of su(2) rather than the Lie algebra $Der(M_3)$ of all derivations of M_3. We have this extra freedom because $Der(M_n)$ is not a module over M_n. All derivations of M_n are inner. This means that every element X of Der(M_n) is of the form $X = adf$ for some f in M_n. In electroweak theory electromagnetic $U_{em}(1)$ whose generator is $\tau_0 + \tau_3$ will remain unbroken. Here τ_0 is identified with $Y + \frac{2}{3}$, where Y is the hypercharge $\frac{\tau_8}{\sqrt{3}}$, and τ_3 and τ_8 are the usual Gell-Mann matrices. Among the above generators of M_3 only the generators of the U-spin subalgebra commute with $\tau_0 + \tau_3$ so we define our derivations as

$$e_a(f) = m_a[\lambda_a, f], \qquad f \in M_3 \tag{1}$$

where a runs through the indices (+, -, 3) and

$$\lambda_\pm = \frac{U_\pm}{\sqrt{2}} \quad , \quad \lambda_3 = U_3 \tag{2}$$

with

$$m_\pm = m \quad , \quad m_3 = \frac{m^2}{M}. \tag{3}$$

Here m and M are two mass scales that have to be introduced into the theory to keep the dimensions correct. The vector space $Der_2(M_n)$ has a Lie algebra structure given by

$$[e_a, e_b] = \sum_c \frac{m_a m_b}{m_c} \, C_{ab}{}^c \, e_c \tag{4}$$

and hence forms a Lie subalgebra of $Der(M_3)$. The structure constants ${C_{ab}}^c$ are

$$ {C_{+-}}^3 = -{C_{-+}}^3 = 1, \qquad {C_{3+}}^- = -{C_{+3}}^- = 1, \qquad {C_{-3}}^+ = -{C_{3-}}^+ = 1, \quad (5) $$

with all others equal to zero.

To define the algebra of forms $\Omega_2^*(M_n)$ over M_n we first set $\Omega_2^0(M_n)$ equal to M_n. Here the subindex 2 refers to the fact that we are using the derivation algebra $\mathrm{Der}_2(M_3)$. Then we define df for $f \in M_n$ by the equation

$$ df(e_a) = e_a(f). \quad (6) $$

The indices are going to be lowered and raised by the metric

$$ g_{ab} = -Tr(\lambda_a \lambda_b). \quad (7) $$

Then we define the set of one forms ${\Omega_2}^1(M_3)$ to be the set of all elements of the form fdg or dgf with f and g in $\mathcal{A}$ subject to the relations $d(fg) = dfg + fdg$. Because of the noncommutativity $\lambda^a d\lambda^b \neq d\lambda^b \lambda^a$, the set $d\lambda^a$ is not a convenient system of generators for $\Omega_2^1(M_n)$. There is a better set of generators completely characterized by the equations

$$ \begin{array}{rcl} \theta_\pm(e_\mp) & = & 1 \quad , \quad \theta_\pm(e_3) = 0, \\ \theta_3(e_\mp) & = & 0 \quad , \quad \theta_3(e_3) = 1. \end{array} \quad (8) $$

These provide the matrix analogue of the dual basis of 1-forms and satisfy the structure equations

$$ d\theta^a = \sum_{b,c} {C_{bc}}^a \, \frac{m_b m_c}{m_a} \, \theta^b \wedge \theta^c. \quad (9) $$

Note that all θ^a's commute with the elements of M_3. From the generators θ^a, we construct the 1-form

$$ \theta = -m_a \lambda_a \theta^a \quad (10) $$

which is going to play an important role in the definition of covariant derivatives and in the study of gauge fields. It satisfies the zero-curvature condition

$$ d\theta + \theta^2 = 0. \quad (11) $$

We can generalize the above formalism to the case where the algebra $\mathcal{A}$ is the tensor product over the complex numbers of a matrix algebras M_n

and the commutative algebra $C^\infty(V)$, $\mathcal{A} = M_n(\mathbf{C}) \otimes C^\infty(V)$. This will be the underlying algebra in our study of the bosonic sector of the electroweak theory. Let us choose a basis $\theta^\alpha_\beta\ dx^\beta$ of $\Omega^1(V)$ over V and let e_a be the Pfaffian derivations dual to θ^α. Set $i = (\alpha, a), 1 \ \leq i \ \leq \ 4+3 = 7$ and introduce $\theta^i = (\theta^\alpha, \theta^a)$ as generators of $\Omega^1(\mathcal{A})$ as a left or right $\mathcal{A}$-module and $e_i = (e_\alpha, e_a)$ as a basis of $\mathrm{Der}_2(\mathcal{A})$ as a direct sum

$$\Omega^1(\mathcal{A}) = \Omega^1_h \oplus \Omega^1_v \tag{12}$$

where

$$\Omega^1_h = \Omega^1(V) \otimes M_n \qquad , \qquad \Omega^1_v = \mathcal{C}^\infty(V) \otimes \Omega^1(M_n). \tag{13}$$

The exterior derivative df of an element f of $\mathcal{A}$ is can be written as the sum horizontal and vertical parts as

$$df = d_h f + d_v f. \tag{14}$$

In what follows we will work in Cartesian coordinates, that is we take $e_\alpha = \partial_\alpha$ and $\theta^\alpha = dx^\alpha$. Hence we have

$$d_h = dx^\alpha \partial_\alpha. \tag{15}$$

The gauge potential, which is an element of $\Omega^1(V)$ for a trivial U(1)-bundle can be generalised to the noncommutative case as an anti-Hermitian element of $\Omega^1(\mathcal{A})$. Let ω be such an element of $\Omega^1(\mathcal{A})$. We can write it then as

$$\omega = A + \theta + \Phi \tag{16}$$

where

$$\begin{aligned} A &= -igA_\alpha \theta^\alpha \quad \in \Omega^1_h(\mathcal{A}) \\ \Phi &= g\phi_a \theta_a \quad \in \Omega^1_v(\mathcal{A}) \end{aligned} \tag{17}$$

and θ is as given by (10). Note that ω is an anti-Hermitian element of $\Omega^1(\mathcal{A})$, that is, $\omega + \omega^* = 0$ as it should be. g is the coupling constant of the theory. ϕ_a are interpreted as the Higgs fields. The gauge transformations of the trivial U(1)-bundle over V are the unitary elements of $C^\infty(V)$. In analogy, we will choose the group of local gauge transformations as the group of unitary elements $\mathcal{U}$ of $\mathcal{A}$, that is the group of invertible elements of $u \in \mathcal{A}$ satisfying $uu^* = 1$. Here * is the *-product induced in $\mathcal{A}$ and $\mathcal{A}$ is considered

as the set of functions on V with values in GL_n. The curvature 2-form Ω and the field strength F are defined as usual:

$$\Omega = d\omega + \omega^2 \qquad , \qquad F = d_h A + A^2. \tag{18}$$

In terms of components and with

$$\Omega = \frac{1}{2}\ \Omega_{ij}\theta^i \wedge \theta^j \quad , \quad F = \frac{1}{2} F_{\alpha\beta}\theta^\alpha \wedge \theta^\beta \tag{19}$$

we find

$$\begin{aligned} \Omega_{\alpha\beta} &= F_{\alpha\beta}, \\ \Omega_{\alpha a} &= g\mathcal{D}_\alpha\phi_a = g(\partial_\alpha\phi_a - ig\ [A_\alpha, \phi_a]), \\ \Omega_{ab} &= g^2[\phi_a, \phi_b] - g\sum_c \frac{m_a m_b}{m_c}\ C_{ab}{}^c\ \phi_c. \end{aligned} \tag{20}$$

We can now write down the usual gauge invariant Yang-Mills Lagrangian density 4-form

$$\mathcal{L} = -\frac{1}{2g^2}\ Tr(\Omega_{ij}\Omega^{ij}). \tag{21}$$

In terms of the components of Ω it becomes

$$\mathcal{L} = -\frac{1}{2g^2}\ Tr(F_{\alpha\beta}F^{\alpha\beta}) + Tr(\mathcal{D}_\alpha\phi_a\ \mathcal{D}^\alpha\phi^a) - V(\phi) \tag{22}$$

where the Higgs potential

$$V(\phi) = -\frac{1}{2} Tr(\Omega_{ab}\Omega^{ab}). \tag{23}$$

From the form of Ω_{ab} in (20) we see that $V(\phi)$ vanishes for values

$$\phi_a = 0, \qquad \phi_a = \frac{m_a}{g}\ \lambda_a. \tag{24}$$

The first choice corresponds to a symmetric vacuum, while the spontaneously broken symmetric phase obtains for the second vacuum configuration above. In the latter case, the second term on the right hand side of (22) becomes

$$\frac{1}{g^2}\ Tr([A_\alpha, m_a\lambda_a][A^\alpha, m_a\lambda^a]) \tag{25}$$

which is quadratic in gauge potentials and hence it gives mass to vector bosons. This means we have a naturally built-in Higgs mechanism.

4 Bosonic Mass Spectrum

To study the scalar masses it is convenient to write the three independent Higgs fields as follows:

$$\phi_+ = \frac{H^\dagger}{\sqrt{2}}, \qquad \phi_- = \frac{H}{\sqrt{2}}, \qquad \phi_3 = \triangle + \frac{m^2}{2Mg}\,(2\tau_0 - 1) \tag{26}$$

By using the metric components (7) we see

$$\phi^+ = -2\phi_-, \quad \phi^- = -2\phi_+, \quad \phi^3 = -2\phi_3 \tag{27}$$

where

$$\begin{aligned} H &= H_+V_+ + H_0U_+, \\ \triangle &= \frac{1}{2}\,(\triangle_0\lambda_0 + \triangle_a\lambda_a). \end{aligned} \tag{28}$$

For the gauge potentials we write

$$A = -igA_\mu dx^\mu = -ig\frac{1}{2}\,(B_\mu\lambda_0 + W_{\mu a}\lambda_a)dx^\mu \tag{29}$$

B and W's are going describe the gauge bosons, while H's and $\triangle$'s the scalar Higgs bosons. Using the field components above we can write the connection 1-form explicitly:

$$\begin{aligned} \omega &= A + \frac{g}{\sqrt{2}}\,H\theta_- + \frac{g}{\sqrt{2}}\,H^*\theta_+ + g\triangle\theta_3 \\ &\quad -\frac{m}{\sqrt{2}}\,U_+\theta_- - \frac{m}{\sqrt{2}}\,U_-\theta_+ + \frac{m^2}{4M}\,(\lambda_0 + \lambda_3)\theta_3. \end{aligned} \tag{30}$$

The next step is to construct the corresponding curvature 2-form Ω with components

$$\begin{aligned} \Omega_{\mu\nu} &= F_{\mu\nu}, \\ \Omega_{\mu+} &= \frac{g}{\sqrt{2}}\,\mathcal{D}_\mu H, \qquad \Omega_{\mu-} = \Omega^*_{\mu+}, \\ \Omega_{\mu 3} &= g\mathcal{D}_\mu\triangle \end{aligned} \tag{31}$$

where

$$\mathcal{D}_\mu = d_h - ig[A_\mu,] \tag{32}$$

and the remaining three terms are

$$\Omega_{+-} = \frac{g^2}{2}\,[H,H^*] - gM\triangle - m^2\lambda_0 + \frac{m^2}{2}, \tag{33}$$
$$\Omega_{+3} = -\frac{g^2}{\sqrt{2}}\,\triangle H \qquad \Omega_{3-} = \Omega_{+3}^*$$

We write out the Lagrangian density and read the Higgs potential

$$\begin{aligned}\frac{1}{g^2}\,V(H,\triangle) &= \frac{1}{8}\left[H^\dagger H - \frac{m^2}{g^2}\right]^2 \\ &+\frac{1}{4}\left[\frac{1}{2}\,H^\dagger H - \frac{M}{g}\,\triangle_0 - \frac{m^2}{g^2}\right]^2 \\ &+\frac{1}{4}\left[\frac{1}{2}\,H^\dagger\sigma_a H - \frac{M}{g}\,\triangle_a\right]^2 \\ &+\frac{1}{8}\,H^\dagger(\triangle_0+\triangle_a\sigma_a)^2 H.\end{aligned} \tag{34}$$

Here H is written as a two-column vector with entries H_+ and H_0 and σ_a are the Pauli spin matrices. The vacuum configuration can be found either directly from the minimum of the above potential (which is a sum of squares) or from (24) to be

$$H_0 = \frac{m}{g} \quad H_+ = 0 \quad \triangle_0 = \triangle_3 = -\frac{m^2}{2Mg} \quad \triangle_{1,2} = 0 \tag{35}$$

where only the electromagnetism survives symmetry breaking.

The mass spectrum of the model is as follows: The W and Z bosons have masses $\frac{m}{\sqrt{2}}\sqrt{1+\frac{m^2}{2M^2}}$ and m respectively as can be calculated from (25). To determine the mass spectrum of the Higgs sector we first write down the linearized field equations for the Higgs fields and then diagonalize the mass matrix [8]. What we have is five heavy scalars with masses converging to $\sqrt{2}M$ in the limit $M \gg m$, three zero-mass scalars referring to Goldstone modes which would be absorbed by weak intermediate bosons to become massive, and one light Higgs boson with mass $\sqrt{2}m$.

These mass relations are valid at the mass scale M, but we must predict the mass values at the electroweak scale $E_Z \sim m$. This is done by considering the renormalization group flow of the coupling constants $g, g^{'}$ and the Higgs self-coupling constant λ down from the scale M to scale m. The relations

$\lambda = \frac{g^2}{4}$ and $g = g^{'}$ imposed at the mass scale M will let us predict the Higgs masses. The relevant renormalisation group equations are [9]

$$16\pi^2 \frac{dg}{dt} = -\frac{19}{6} g^3 \tag{36}$$

$$16\pi^2 \frac{dg'}{dt} = \frac{41}{6} g'^3 \tag{37}$$

$$16\pi^2 \frac{d\lambda}{dt} = 24\lambda^2 - 3\lambda(3g^2 + g'^2) + \frac{3}{8}\ [2g^4 + (g^2 + g'^2)^2] \tag{38}$$

We first solve (36) and (37) at arbitrary scales μ subject to the above constraints at the mass scale M. Then we fix the measured values of g and g' at the electroweak scale $\mu = E_Z = 91GeV$ which are $g(E_Z) = 0.4234$ and $g'(E_Z) = 0.1278$. This requires $M \sim 5 \times 10^{20} GeV$. The remaining equation (38) can now be solved numerically by feeding in the solutions of equations (36) and (37), leading to the result $\lambda(E_Z) = 0.14$. Finally we use the standard model relation

$$\frac{m_H^2(\mu)}{m_Z^2(\mu)} = \frac{8\lambda(\mu)}{g^2(\mu) + g'^2(\mu)} \tag{39}$$

and noting that at scales M, $m_H = m_Z = \sqrt{2}m$ and $g^2(M) = g'^2(M) = 4\lambda(M) = 0.49$ determine the Higgs boson mass at the electroweak scales. To the extent that radiative corrections can be neglected [10], we find

$$m_H(E_Z) \sim 130 GeV. \tag{40}$$

5 Concluding Comments

The noncommutative electroweak model of Balakrishna, Gürsey and Wali is minimal in its construction and the bosonic mass spectrum it predicts is realistic. The model could be generalised in several directions. One line of development would concern coupling of spinorial matter and to seek supersymmetrization [11]. On the other hand it is emphasised very often in the literature [12, 13] that approaches based on noncommutative geometries not only yield promising reformulations of the standard model but would also be relevant to gravitation. The invariant tensor formulation of the model given here is especially well suited for both of these lines of investigation.

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